Lesson 6: L'Hospital's Rule

Topic 4.7: Using L'Hospital's Rule for Determining Limits of Indeterminate Forms

L'Hospital's Rule is a great tool for finding limits that are often difficult to evaluate. It can also help with determining limits at infinity for "asymptotic behavior." We will use it for curve sketching in the next section, as well.

When evaluating limits by direct substitution, there are seven expressions known as **indeterminate form**. These expressions are

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 1^{\infty}, 0^0, \infty - \infty, \infty^0$$

Consider the limit of a quotient: $\lim_{x \to c} \frac{f(x)}{g(x)}$.

L'Hospital's Rule states that *when $f(x)/g(x)$ has an indeterminate form of the type $0/0$ or ∞/∞ at $x = c$, then we can replace $f(x)/g(x)$ by the quotient of the derivatives $f'(x)/g'(x)$.*

L'Hospital's Rule

Assume $f(x)$ and $g(x)$ are differentiable on an open interval (a, b) containing c. Also assume that $g'(x) \neq 0$ for all x in (a, b); except possibly at c itself. Then, if

$$\lim_{x \to c} f(x) = 0 \text{ and } \lim_{x \to c} g(x) = 0 \text{ ; then}$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

If the limit on the right exists or is infinite (∞ or $-\infty$). This conclusion also holds if $f(x)$ and $g(x)$ are differentiable for x near (but not equal to) c and

$$\lim_{x \to c} f(x) = \pm\infty \text{ and } \lim_{x \to c} g(x) = \pm\infty$$

This rule is also valid for one-sided limits.

A Word of Caution:

In order to be able to use L'Hospital's Rule; it is important to note that these conditions are met:

1. $g'(c) \neq 0$; If the denominator is zero then the limit is undefined.
2. No oscillating functions, as x approaches c, the limit must approach a single y-value.
3. $\lim_{x \to c} \left(\frac{f(x)}{g(x)}\right) \neq \pm\infty$; if the limit approaches infinity as x approaches c, the limit is undefined.

Communication in mathematics is an important skill. Let's turn our discussion to the proper notation and protocol for using L'Hospital's Rule correctly on the newly redesigned AP Calculus exams before we practice.

Proper Notation and Protocol

In order to use L'Hospital's Rule on the AP Calculus Exam® students should show proper procedural steps and notation. To master the process be sure to . . .

1. Explicitly show the limit of the numerator is 0, ∞, or $-\infty$.
2. Explicitly show the limit of the denominator is 0, ∞, or $-\infty$.
3. You may want to state that the limit is indeterminate form because they are both 0 or $\pm\infty$.
4. By L'Hospital's Rule, write the limit as $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$
5. Find the limit using proper limit notation throughout.

Type I: Simple Polynomials and Powers

EX #1: $\lim_{x \to 5} \frac{2x^2 - 9x - 5}{x - 5}$

EX #2: $\lim_{x \to -5} \frac{\sqrt{4 - x} - 3}{x + 5}$

Type 2: Repeated Use of L'Hospital's Rule

EX #3: $\lim_{x \to -\infty} \frac{x^2}{e^{-x}}$

EX #4: $\lim_{x \to -2} \frac{x^3 + x^2 - 8x - 12}{x^3 + 8x^2 + 20x + 16}$

Type 3: Trigonometry and Transcendental Functions

EX #5: $\lim_{x \to 0} \frac{\sin x(1 - \cos x)}{x^2}$

EX #6: $\lim_{x \to 1} \frac{\ln x - x + 1}{e^x - e}$

Type 4: L'Hospital's Rule for $\pm\frac{\infty}{\infty}$

EX #7: $\lim_{x \to \infty} \frac{9x^2 + 10x - 3}{12x^2 - 4}$

EX #8: $\lim_{x \to \infty} \frac{4x^2 - 3x + 2}{e^{2x} + \ln x}$

Type 5: AP Style Applications of L'Hospital's Rule

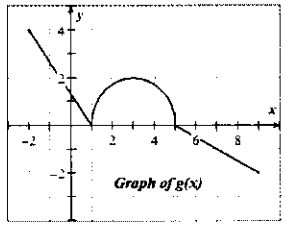
EX #9: The table below gives values of a twice-differentiable function $y = f(x)$. Use the table to find the limit, if it exists. Justify your answer.

$$\lim_{x \to -2} \frac{f(3x + 2)}{x^2 - 4}$$

x	-4	-1	2
$f(x)$	0	3	0
$f'(x)$	8	-2	6

EX #10: Use the graph of $y = g(x)$, shown below to find the following limit, if possible. Use proper notation to justify your findings.

$$\lim_{x \to 3} \frac{g(x)}{x-3}$$



Graph of g(x)

EX #11: The differential equation for the curve $y = h(x)$ is given by $\frac{dy}{dx} = \frac{3x}{y+2}$. Given that h is twice-differentiable and $h(3) = -1$, find:

$$\lim_{x \to 3} \frac{h(x) + x - 2}{\ln(7-2x)}$$

EX #12: Find the limit, if possible. Give a reason for your answer.

$$\lim_{x \to \infty} \frac{2^x}{x^{99}}$$