Lesson 5: Linear Approximation

Topic 4.6: Approximating Values of a Function Using Local Linearity and Linearization

In Unit 2 we briefly looked at the idea of local linearity, when zooming in on a function, we discovered as we zoom in on a function $y = f(x)$, the straight line that appears at a given point $x = a$ has a slope equal to the derivative $f'(a)$, and the equation for the tangent line at the given point was defined as:

$$f(x) = f(a) + f'(a)(x - a)$$

In this lesson, we will expand our investigation for a deeper understanding.

The Tangent Line Approximation

Suppose f is a differentiable function at a point $x = a$. Then, for values of x near a, the tangent line approximation to $f(x)$ is

$$f(x) \approx f(a) + f'(a)(x - a)$$

We will define the expression $f(a) + f'(a)(x - a)$ as the ***local linearization of f near $x = a$***.

If you consider that point a is fixed, then $f(a)$ and $f'(a)$ are constant.

So, $f(x) \approx L(x) = f(a) + f'(a)(x - a)$ if x is close to a.

Graphically, linear approximation is $\Delta f \approx f'(a)\,\Delta x$, where $\Delta f = f(a + \Delta x) - f(a)$

The Linear Approximation Error

The **error, $E(x)$**, in the linear approximation is simply the vertical "gap" between the actual curve and the tangent line. The error can be calculated by:

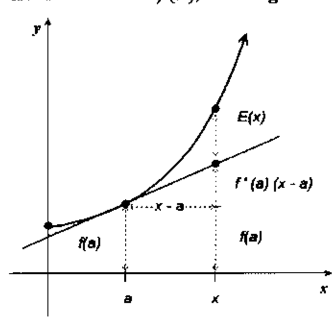
Error = |Exact − Approximation|

$$E(x) = |\Delta f - f'(a)\,\Delta x|$$

$$\text{Percentage Error} = \left|\frac{\text{exact} - \text{approximation}}{\text{exact}}\right| \times 100\%$$

Visual Vocabulary

EX #1: It can be shown that the best linear approximation to f near a is the tangent line approximation. Label the true value of $f(x)$, the tangent line, and the approximation.



Procedure for Finding a Linear Approximation for f(x)

So, if your head is spinning with this new information, let's summarize what we will be focused upon for the remainder of the lesson. When you need to write a linear approximation for $f(x)$ centered at a given value, say $x = c$, and you need to approximate $f(x)$ at a value that is in the neighborhood of $x = c$, let's call this new value $x = a$, then proceed as follows:

1. Write the **point-slope equation** for the tangent line at $(c, f(c))$.
2. Isolate y and **rename** this equation as $L(x)$.
3. **Substitute** your value for $x = a$ in the equation, $L(x)$. Be sure to write the notation correctly:
$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$
4. If you need to explain whether or not $L(a)$ is an **overestimate** or an **underestimate**, consider the behavior of the function in your local neighborhood.
   - ❑ If the function is **concave up** at $x = c$, $L(a)$ is an **underestimation**.
   - ❑ If the function is **concave down** at $x = c$, $L(a)$ is an **overestimation**.
5. When **finding the error** or accuracy of an approximation, use your calculator. Find the ***absolute value of the difference between the exact amount and your approximation***. Then, the error should be stated as being less than 10^{-p}, where p is the number of zeros in your result.
   For example: $|\text{Error}| < 10^{-3}$ would be the reported error for a result such as, 0.000147
6. If you need to find the Percentage Error = $\left|\frac{\text{exact} - \text{approximation}}{\text{exact}}\right| \times 100\%$

EX #2: The function f is twice differentiable where $f(3) = -2$, $f'(3) = 5$ and $f''(3) = 1$. Write the equation of the tangent line. Then, use the equation to find the value of $f(3.1)$.

Linear approximations will do a great job of estimating values for $f(x)$, provided you stay in the small "neighborhood" near $x = a$. The farther away from $x = a$ we move, the worse the approximation becomes. While there is no easy method to determine how near or far away we can move from our point of tangency, we can predict the amount of error to give us an idea of how "good" the approximation models the actual value.

Let's look at a famous linear approximation that is used to describe vibrations, pendulum motion, and is used as a method to simplify formulas in optics.

EX #3: Consider $y = \sin x$ at $x = \frac{\pi}{3}$.

We know the ordered pair corresponding to this angle is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

A. Since the angle $\frac{\pi}{3} \approx 1.0471975\ldots$ Let's write the linear approximation for $x = 1.05$.

B. Find the error of your approximation.

EX #4: Find a linear approximation to $y = 2^x$ at $x = 3$. Use the tangent line to estimate $x = 2.97$

EX #5: Approximate $(2.01)^5$. Use your calculator to find the accuracy of the approximation. Is the linearization an overestimate or underestimate? Explain.

EX #6: Find the linear approximation to $g(x) = \sqrt[3]{x}$ at $x = 8$. Use the linear approximation to approximate the value of $\sqrt[3]{10}$ and $\sqrt[3]{16}$. Compare the approximate values to the exact values. Find the error of your approximations.

EX #7: Let f be a function that is differentiable for all real numbers. The table below gives the values of $f(x)$ and its derivative $f'(x)$ for selected values in the interval $[-1.0, 1.0]$. The function is **concave up** in this closed interval.

x	-1.0	-0.8	-0.6	−0.3	0	0.3	0.6	0.8	1.0
$f(x)$	-12	-36	-51	-64	-87	-73	−67	−19	5
$f'(x)$	-8	-5	-2	−1	0	2	3	6	14

A. Write an equation of the line tangent to the graph of $f(x)$ where $x = -0.6$.

B. Use this line to approximate the value of $f(-0.5)$.

C. Is this approximation greater or less than the actual value of $f(-0.5)$ and give a reason.

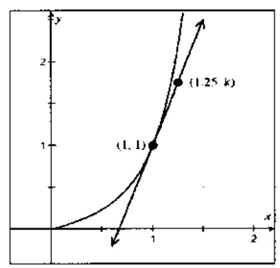
EX #8: Write the linear approximation equation for $f(x) = \frac{1}{x-1}$ at $x = -1$.

A. Find an approximation for $f(-1.2)$

B. Calculate the error of the approximation.

EX #9: Given the implicit curve, $x^2 + \ln\left(\frac{x}{y}\right) = 1$;

A. Write the linear approximation of the tangent line at $x = 1$.



B. Using the tangent line to the curve at $x = 1$, find the number k such that the point $(1.25, k)$ is on the curve.

C. Calculate the error of the approximation.